

Application of the B -Determining Equations Method to One Problem of Free Turbulence^{*}

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Abstract. A three-dimensional model of the far turbulent wake behind a self-propelled body in a passively stratified medium is considered. The model is reduced to a system of ordinary differential equations by a similarity reduction and the B -determining equations method. The system of ordinary differential equations satisfying natural boundary conditions is solved numerically. The solutions obtained here are in close agreement with experimental data.

Key words: turbulence; far turbulent wake; B -determining equations method

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1 Introduction

Most flows occurring in nature and engineering practice are turbulent (see, e.g., [10, 20, 22]). Semiempirical models of turbulence are widely used in the modeling of turbulent flows [16, 21, 27]. However, there are only a few analytical results on such models (see, e.g., [2, 11]).

The far turbulent wake behind an axisymmetric body in a stratified medium is an example of a free shear flow. Sufficiently complete experimental data on the dynamics of turbulent wakes generated by moving bodies in stratified fluids were obtained by Lin and Pao [17].

The far turbulent wake behind an axisymmetric towed body in a linearly stratified medium was numerically simulated in [9]. Chernykh et al. [6] carried out the numerical simulation of the dynamics of turbulent wakes in a stable stratified medium based on hierarchy of second order closure models. Calculation results obtained in [6, 9] are in close agreement with experimental data [17].

Similarity solutions for several turbulence models were constructed in [7, 13, 14, 15]. In the current paper we consider three-dimensional semiempirical model of the far turbulent wake behind an axisymmetrical self-propelled body in a passively stratified medium (see [6, 4, 25] and the references therein).

This paper is organized as follows. In Section 3 we determine the most general continuous classical symmetry group of the model and obtain the similarity reduction of the model. In Section 4 we use the B -determining equations (BDEs) method [1, 12] to transform the reduced system into a system of ordinary differential equations (ODEs).

In the last section, we consider a boundary value problem for the system of ODEs. We use the modified shooting method and the asymptotic expansion of the solution in the vicinity of the singular point to solve this problem. Finally, computational results are given.

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2 Model

The following three-dimensional semiempirical model of turbulence was constructed in [4, 6, 25] to calculate characteristics of the far turbulent wake behind an axisymmetric self-propelled body in a passively stratified medium:

$$U_0 \frac{\partial e}{\partial x} = \frac{\partial}{\partial y} C_e \frac{e^2}{\varepsilon} \frac{\partial e}{\partial y} + \frac{\partial}{\partial z} C_e \frac{e^2}{\varepsilon} \frac{\partial e}{\partial z} - \varepsilon, \quad (1)$$

$$U_0 \frac{\partial \varepsilon}{\partial x} = \frac{\partial}{\partial y} C_\varepsilon \frac{e^2}{\varepsilon} \frac{\partial \varepsilon}{\partial y} + \frac{\partial}{\partial z} C_\varepsilon \frac{e^2}{\varepsilon} \frac{\partial \varepsilon}{\partial z} - C_{\varepsilon 2} \frac{\varepsilon^2}{e}, \quad (2)$$

$$U_0 \frac{\partial \langle \rho_1 \rangle}{\partial x} = \frac{\partial}{\partial y} C_\rho \frac{e^2}{\varepsilon} \frac{\partial \langle \rho_1 \rangle}{\partial y} + \frac{\partial}{\partial z} C_\rho \frac{e^2}{\varepsilon} \frac{\partial \langle \rho_1 \rangle}{\partial z} - \frac{\partial}{\partial z} C_\rho \frac{e^2}{\varepsilon}, \quad (3)$$

$$U_0 \frac{\partial \langle \rho'^2 \rangle}{\partial x} = \frac{\partial}{\partial y} C_{1\rho} \frac{e^2}{\varepsilon} \frac{\partial \langle \rho'^2 \rangle}{\partial y} + \frac{\partial}{\partial z} C_{1\rho} \frac{e^2}{\varepsilon} \frac{\partial \langle \rho'^2 \rangle}{\partial z} + 2C_\rho \frac{e^2}{\varepsilon} \frac{\partial \langle \rho_1 \rangle^2}{\partial y} + 2C_\rho \frac{e^2}{\varepsilon} \left(\frac{\partial \langle \rho_1 \rangle}{\partial z} - 1 \right)^2 - C_T \frac{\langle \rho'^2 \rangle \varepsilon}{e}, \quad (4)$$

where e is the turbulent kinetic energy, ε is the kinetic energy dissipation rate, $\langle \rho_1 \rangle$ is the averaged density defect, and $\langle \rho'^2 \rangle$ is the density fluctuation variance. All the unknown functions depend on x , y , and z . The quantities $C_e = 0.136$, $C_\varepsilon = C_e/\delta$, $\delta = 1.3$, $C_{\varepsilon 2} = 1.92$, $C_\rho = 0.208$, $C_{1\rho} = 0.087$, $C_T = 1.25$ are generally accepted empirical constant [8, 21]. U_0 is the free stream velocity. The marching variable x in the equations (1)–(4) acts as time.

This model is based on the three-dimensional parabolized system of averaged Navier–Stokes equations in the Oberbeck–Boussinesq approximation (see [5, 26])

$$U_0 \frac{\partial U_d}{\partial x} + V \frac{\partial U_d}{\partial y} + W \frac{\partial U_d}{\partial z} = \frac{\partial}{\partial y} \langle u'v' \rangle + \frac{\partial}{\partial z} \langle u'w' \rangle, \quad (5)$$

$$U_0 \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \langle p_1 \rangle}{\partial y} - \frac{\partial}{\partial y} \langle v'^2 \rangle - \frac{\partial}{\partial z} \langle v'w' \rangle, \quad (6)$$

$$U_0 \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \langle p_1 \rangle}{\partial z} - \frac{\partial}{\partial y} \langle v'w' \rangle - \frac{\partial}{\partial z} \langle w'^2 \rangle - g \frac{\langle \rho_1 \rangle}{\rho_0}, \quad (7)$$

$$U_0 \frac{\partial \langle \rho_1 \rangle}{\partial x} + V \frac{\partial \langle \rho_1 \rangle}{\partial y} + W \frac{\partial \langle \rho_1 \rangle}{\partial z} + W \frac{\partial \rho_s}{\partial z} = -\frac{\partial}{\partial y} \langle v'\rho' \rangle - \frac{\partial}{\partial z} \langle w'\rho' \rangle, \quad (8)$$

$$\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{\partial U_d}{\partial x}, \quad (9)$$

where $U_d = U_0 - U$ is the defect of the averaged longitudinal velocity component; U , V and W are the mean flow velocity component along x -, y - and z -axes, respectively; $\langle p_1 \rangle$ is the deviation from the hydrostatic pressure due to stratification $\rho_s(z)$; g is the gravity acceleration; $\langle \rho_1 \rangle$ is the averaged density defect: $\rho_1 = \rho - \rho_s$; $\rho_s = \rho_s(z)$ is the undisturbed fluid density assumed to be linear: $\rho_s(z) = \rho_0(1 - az)$, $a > 0$ is a constant; the prime indicates the pulsating components; $\langle \rangle$ indicates averaging.

In [9, 26] the Reynolds stress tensor components $\langle u'_i u'_j \rangle$, the turbulent flows $\langle u'_i \rho' \rangle$, and the density fluctuation variance $\langle \rho'^2 \rangle$ are defined by the algebraic relations [21]. Since the flow in the far turbulent wake is considered, these relations are simplified as follows

$$\langle u'v' \rangle = \frac{1 - c_2}{c_1} \frac{e \langle v'^2 \rangle}{\varepsilon} \frac{\partial U_d}{\partial y} = K_y \frac{\partial U_d}{\partial y}, \quad (10)$$

$$\langle u'w' \rangle = \frac{(1 - c_2)e \langle w'^2 \rangle - \frac{(1 - c_3)(1 - c_{2T})}{c_{1T}} \frac{g}{\rho_0} \frac{e^2}{\varepsilon} \langle w'\rho' \rangle}{c_1 \varepsilon \left(1 - \frac{1 - c_3}{c_{1T}} \frac{g}{\rho_0} \frac{e^2}{\varepsilon^2} \frac{\partial \langle \rho \rangle}{\partial z} \right)} \frac{\partial U_d}{\partial z} = K_z \frac{\partial U_d}{\partial z}, \quad (11)$$

$$\langle v'^2 \rangle = \frac{2}{3}e \left(1 - \frac{1-c_2}{c_1} \frac{P}{\varepsilon} - \frac{1-c_2}{c_1} \frac{G}{\varepsilon} \right), \quad (12)$$

$$\langle w'^2 \rangle = \frac{2}{3}e \left(1 - \frac{1-c_2}{c_1} \frac{P}{\varepsilon} + 2 \frac{1-c_2}{c_1} \frac{G}{\varepsilon} \right), \quad (13)$$

$$\langle \rho'^2 \rangle = \frac{2}{c_T} \frac{e}{\varepsilon} \langle w' \rho' \rangle \frac{\partial \langle \rho \rangle}{\partial z}, \quad (14)$$

$$-\langle u' \rho' \rangle = \frac{1}{c_{1T}} \frac{e}{\varepsilon} \left(\langle u' w' \rangle \frac{\partial \langle \rho \rangle}{\partial z} + (1-c_{2T}) \langle w' \rho' \rangle \frac{\partial \langle U \rangle}{\partial z} \right), \quad (15)$$

$$-\langle v' \rho' \rangle = \frac{1}{c_{1T}} \frac{e}{\varepsilon} \langle v'^2 \rangle \frac{\partial \langle \rho \rangle}{\partial y} = K_{\rho y} \frac{\partial \langle \rho \rangle}{\partial y}, \quad (16)$$

$$\begin{aligned} -\langle w' \rho' \rangle &= \frac{e}{c_{1T} \varepsilon} \left(\langle w'^2 \rangle \frac{\partial \langle \rho \rangle}{\partial z} + (1-c_{2T}) \frac{g}{\rho_0} \langle \rho'^2 \rangle \right) \\ &= \frac{e \langle w'^2 \rangle}{c_{1T} \varepsilon \left(1 - 2 \frac{1-c_{2T}}{c_{1T} c_{2T}} \frac{g}{\rho_0} \frac{e^2}{\varepsilon^2} \frac{\partial \langle \rho \rangle}{\partial z} \right)} \frac{\partial \langle \rho \rangle}{\partial z} = K_{\rho z} \frac{\partial \langle \rho \rangle}{\partial z}, \end{aligned} \quad (17)$$

$$P = \left(\langle u' v' \rangle \frac{\partial U_d}{\partial y} + \langle u' w' \rangle \frac{\partial U_d}{\partial z} \right), \quad (18)$$

$$G = -\frac{1}{\rho_0} \langle w' \rho' \rangle g. \quad (19)$$

Differential transport equations [21] are used in [4, 6, 25] to determine the kinetic turbulent energy e , the kinetic energy dissipation rate ε , and the shear Reynolds stress $v'w'$:

$$U_0 \frac{\partial e}{\partial x} + V \frac{\partial e}{\partial y} + W \frac{\partial e}{\partial z} = \frac{\partial}{\partial y} K_{ey} \frac{\partial e}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial e}{\partial z} + P + G - \varepsilon, \quad (20)$$

$$U_0 \frac{\partial \varepsilon}{\partial x} + V \frac{\partial \varepsilon}{\partial y} + W \frac{\partial \varepsilon}{\partial z} = \frac{\partial}{\partial y} K_{\varepsilon y} \frac{\partial \varepsilon}{\partial y} + \frac{\partial}{\partial z} K_{\varepsilon z} \frac{\partial \varepsilon}{\partial z} + c_{\varepsilon 1} \frac{\varepsilon}{e} (P + G) - c_{\varepsilon 2} \frac{\varepsilon^2}{e}, \quad (21)$$

$$\begin{aligned} U_0 \frac{\partial \langle v' w' \rangle}{\partial x} + V \frac{\partial \langle v' w' \rangle}{\partial y} + W \frac{\partial \langle v' w' \rangle}{\partial z} &= \frac{\partial}{\partial y} K_{ey} \frac{\partial \langle v' w' \rangle}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial \langle v' w' \rangle}{\partial z} \\ &+ (c_2 - 1) \left(\langle v'^2 \rangle \frac{\partial W}{\partial y} + \langle w'^2 \rangle \frac{\partial V}{\partial z} \right) + (1 - c_3) \frac{g}{\rho_0} \langle v' \rho' \rangle - c_1 \frac{\varepsilon}{e} \langle v' w' \rangle. \end{aligned} \quad (22)$$

The turbulent viscosity coefficients in these equations are $K_{ey} = K_y$, $K_{ez} = K_z$, $K_{\varepsilon y} = K_{ey}/\delta$, $K_{\varepsilon z} = K_{ez}/\delta$. The model (1)–(4) is an analogue of the equations (5)–(22) (for the diffusion approximation in a homogeneous fluid $V = 0$, $W = 0$, and $g = 0$). In what follows, we assume that the free stream velocity U_0 equals unity.

3 Similarity solution

The infinitesimal symmetry group [18, 19] of the model (1)–(4) is spanned by the eight vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}, & X_3 &= \frac{\partial}{\partial z}, & X_4 &= \frac{\partial}{\partial \langle \rho_1 \rangle}, & X_5 &= -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + y \frac{\partial}{\partial \langle \rho_1 \rangle}, \\ X_6 &= y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + 2e \frac{\partial}{\partial e} + 2\varepsilon \frac{\partial}{\partial \varepsilon} + \langle \rho_1 \rangle \frac{\partial}{\partial \langle \rho_1 \rangle} + 2 \langle \rho'^2 \rangle \frac{\partial}{\partial \langle \rho'^2 \rangle}, \\ X_7 &= x \frac{\partial}{\partial x} - 2e \frac{\partial}{\partial e} - 3\varepsilon \frac{\partial}{\partial \varepsilon}, & X_8 &= (\langle \rho_1 \rangle - z) \frac{\partial}{\partial \langle \rho_1 \rangle} + 2 \langle \rho'^2 \rangle \frac{\partial}{\partial \langle \rho'^2 \rangle}. \end{aligned}$$

Available experimental data [10, 17] and numerical calculations [9, 20, 27] show that the flow in the far turbulent wake can be considered to be close to a self-similar flow. We therefore consider the linear combination of scaling vector fields X_6 and X_7

$$Z = x \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + \alpha z \frac{\partial}{\partial z} + 2(\alpha - 1)e \frac{\partial}{\partial e} + (2\alpha - 3)\varepsilon \frac{\partial}{\partial \varepsilon} + \alpha \langle \rho_1 \rangle \frac{\partial}{\partial \langle \rho_1 \rangle} + 2\alpha \langle \rho'^2 \rangle \frac{\partial}{\partial \langle \rho'^2 \rangle}.$$

The similarity variables associated with the infinitesimal generator Z are

$$\begin{aligned} \xi &= \frac{y}{x^\alpha}, & \eta &= \frac{z}{x^\alpha}, & e &= x^{2\alpha-2}E(\xi, \eta), & \varepsilon &= x^{2\alpha-3}G(\xi, \eta), \\ \langle \rho_1 \rangle &= x^\alpha H(\xi, \eta), & \langle \rho'^2 \rangle &= x^{2\alpha}R(\xi, \eta), \end{aligned}$$

where E , G , H , and R are arbitrary functions. According to physical considerations (the influence of gravity is neglected in this problem), the functions E and G must be of the form

$$E(\xi, \eta) = E\left(\sqrt{\xi^2 + \eta^2}\right), \quad G(\xi, \eta) = G\left(\sqrt{\xi^2 + \eta^2}\right). \quad (23)$$

We obtain the reduced system by introducing similarity variables. Using (23) and changing to polar coordinates $\xi = r \cos \phi$, $\eta = r \sin \phi$ the reduced system becomes

$$C_e \frac{E}{G} \left(EE'' + 2E'^2 - \frac{E}{G} E' G' + \frac{E}{r} E' \right) + \alpha r E' + 2(1 - \alpha)E - G = 0, \quad (24)$$

$$C_\varepsilon \frac{E}{G} \left(EG'' - \frac{E}{G} G'^2 + 2E' G' + \frac{E}{r} G' \right) + \alpha r G' + (3 - 2\alpha)G - C_{\varepsilon 2} \frac{G^2}{E} = 0, \quad (25)$$

$$\begin{aligned} C_\rho \frac{E^2}{G} \left(H_{rr} + \frac{1}{r^2} H_{\phi\phi} \right) + \left(C_\rho \frac{E}{G} \left(2E' - \frac{E}{G} G' + \frac{E}{r} \right) + \alpha r \right) H_r - \alpha H \\ + C_\rho \frac{E}{G} \left(\frac{E}{G} G' - 2E' \right) \sin \phi = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} C_{1\rho} \frac{E^2}{G} \left(R_{rr} + \frac{1}{r^2} R_{\phi\phi} \right) + \left(C_{1\rho} \frac{E}{G} \left(\frac{E}{r} + 2E' - \frac{E}{G} G' \right) + \alpha r \right) R_r - \left(C_T \frac{E}{G} + 2\alpha \right) R \\ + 2C_\rho \frac{E^2}{G} \left(H_r^2 + \frac{1}{r^2} H_\phi^2 \right) + 2C_\rho \frac{E^2}{G} - 4C_\rho \frac{E^2}{G} \left(H_r \sin \phi + H_\phi \frac{\cos \phi}{r} \right) = 0, \end{aligned} \quad (27)$$

where $E = E(r)$, $G = G(r)$, $H = H(r, \phi)$, and $R = R(r, \phi)$. Here, and throughout, subscripts denote derivatives, so $H_r = \partial H / \partial r$, etc.

Lie's classical method do not provide solution of the reduced system agreed with experimental data. We therefore use the BDEs method.

4 BDEs method

The concept of BDEs of a system of partial differential equations (PDEs) was introduced in [1, 12]. Consider a scalar PDE

$$\Omega(\mathbf{x}, u, u^1, u^2, \dots) = 0, \quad (28)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n independent variables, u denotes the dependent variable, and

$$u_k = \frac{\partial u^k}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}$$

denotes the set of coordinates corresponding to all k -th order partial derivatives of u with respect to \mathbf{x} . In the BDEs method, an extension of the classical symmetry determining relations is made by incorporating an additional factor $B(\mathbf{x}, u, u^1, u^2, \dots)$. For a scalar PDE (28), BDE is

$$h \frac{\partial \Omega}{\partial u} + \sum_{|\alpha| \geq 1} D^\alpha(h) \frac{\partial \Omega}{\partial u_\alpha} + Bh|_{\Omega=0} = 0, \quad (29)$$

where h is a function of $\mathbf{x}, u, u^1, u^2, \dots$; $D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$, D_{x_i} is the total x_i derivative; α is a multi-index. Equality (29) must hold for all solutions of (28).

Now we use the BDEs method to reduce (26) and (27) to some ODEs. Consider more general equation than (26)

$$H_{\phi\phi} + r^2 H_{rr} + A(r)H_r + B(r)H + C(r)\sin\phi = 0, \quad (30)$$

where $A(r)$, $B(r)$, and $C(r)$ are arbitrary functions. BDE corresponding to (30) is

$$D_\phi^2 h + r^2 D_r^2 h + b_1(r, \phi) D_r h + b_2(r, \phi) h = 0. \quad (31)$$

Here, and throughout D_ϕ , D_r are the operators of total differentiation with respect to ϕ and r . The function h may depend on r , ϕ , H and derivatives of H . The functions $b_1(r, \phi)$ and $b_2(r, \phi)$ are to be determined together with the function h . Note that if we let in (31)

$$b_1(r, \phi) = A(r), \quad b_2(r, \phi) = B(r),$$

we obtain the classical determining equation [18, 19].

For simplicity, we assume that a solution of (31) is independent of r and partial derivatives of H with respect to r

$$h = H_{\phi\phi} + h_1(\phi, H, H_\phi). \quad (32)$$

Substituting (32) into (31) gives a polynomial equation for derivatives of the fourth order. This polynomial must identically vanish. We can express the derivatives $H_{rr\phi\phi}$, $H_{\phi\phi\phi\phi}$, $H_{r\phi\phi}$, $H_{\phi\phi\phi}$, and $H_{\phi\phi}$ using (30). The coefficient of H_{rrr} implies $b_1(r, \phi) = A(r)$.

As a result, the left side of (31) is a polynomial in H_{rr} and $H_{r\phi}$. Collecting similar terms we obtain

$$\begin{aligned} 2(A(r)H_r + B(r)H + C(r)\sin\phi)h_{1_{H_\phi H_\phi}} - 2H_\phi h_{1_{HH_\phi}} - 2h_{1_{\phi H_\phi}} + B(r) - b_2(r, \phi) &= 0, \\ h_{1_{H_\phi H_\phi}} &= 0, \quad h_{1_{HH_\phi}} = 0. \end{aligned}$$

Hence

$$h_1(\phi, H, H_\phi) = h_2(\phi)H_\phi + h_3(\phi, H), \quad b_2(r, \phi) = B(r) - 2h'_2(\phi).$$

Substituting the functions b_1 , b_2 , and h_1 into the left side of (31) we obtain the polynomial with respect to H_r and H_ϕ . This polynomial must identically vanish. Collecting similar terms we have

$$\begin{aligned} (B(r)H + C(r)\sin\phi)h_{3_H} - h_{3_{\phi\phi}} + (2h'_2 - B(r))h_3 + C(r)(h_2 \cos\phi - \sin\phi) &= 0, \\ h_{3_{HH}} &= 0, \quad 2h_{3_{\phi H}} + h''_2 - 2h'_2 h_2 = 0. \end{aligned}$$

Thus,

$$h_3(\phi, H) = \left(\frac{1}{2}h_2^2 - \frac{1}{2}h'_2 + h_4 \right) H, \quad h'_2 - h_2^2 - 2\cot\phi h_2 + 2(1 - h_4) = 0, \quad (33)$$

where h_4 is an arbitrary constant.

Clearly, that the Riccati equation (33) has the partial solution

$$h_2 = \tan \phi$$

for $h_4 = 1/2$.

Thus we find the solution of (31)

$$h = H_{\phi\phi} + H_{\phi} \tan \phi.$$

The corresponding differential constraint $h = 0$ has the general solution

$$H = H_1(r) \sin \phi + H_2(r), \quad (34)$$

where H_1 and H_2 are arbitrary functions.

Next we use (34) and consider the equation (27) in more general form

$$R_{\phi\phi} + r^2 R_{rr} + K(r) R_r + L(r) R + M(r) \sin^2 \phi + N(r) \sin \phi + P(r) = 0, \quad (35)$$

where $K(r)$, $L(r)$, $M(r)$, $N(r)$, and $P(r)$ are arbitrary functions. The BDEs method applied to the equation (35) gives rise to the following results:

$$\begin{aligned} N(r) &= 0, & b_1(r, \phi) &= K(r), & b_2(r, \phi) &= L(r) - 8 \sin^{-2} 2\phi, \\ h &= R_{\phi\phi} - 2R_{\phi} \cot 2\phi. \end{aligned}$$

Integrating differential constraint $h = 0$ corresponding to the BDE solution (34), we find

$$R = R_1(r) \sin^2 \phi + R_2(r), \quad (36)$$

where $R_1(r)$ and $R_2(r)$ are arbitrary functions. Using (34) and (36) we obtain the following corollary in terms of variables x , y , and z .

Corollary 1. *The following expressions for the unknown functions*

$$e = x^{2\alpha-2} E \left(\frac{\sqrt{y^2 + z^2}}{x^\alpha} \right), \quad \varepsilon = x^{2\alpha-3} G \left(\frac{\sqrt{y^2 + z^2}}{x^\alpha} \right), \quad (37)$$

$$\langle \rho_1 \rangle = z H \left(\frac{\sqrt{y^2 + z^2}}{x^\alpha} \right), \quad \langle \rho'^2 \rangle = z^2 R_1 \left(\frac{\sqrt{y^2 + z^2}}{x^\alpha} \right) + x^{2\alpha} R_2 \left(\frac{\sqrt{y^2 + z^2}}{x^\alpha} \right) \quad (38)$$

allow us to reduce the model (1)–(4) to a system of ODEs.

Indeed, substituting (37) and (38) into (1)–(4) we obtain

$$\frac{E^2 E''}{G} - \frac{E^2 E'}{G} \left(\frac{G'}{G} - 2 \frac{E'}{E} - \frac{1}{\tau} \right) - \frac{1}{C_e} (2(\alpha - 1)E + G - \alpha \tau E') = 0, \quad (39)$$

$$\frac{E^2 G''}{G} - \frac{E^2 G'}{G} \left(\frac{G'}{G} - 2 \frac{E'}{E} - \frac{1}{\tau} \right) - \frac{1}{C_e} \left((2\alpha - 3)G + C_{e2} \frac{G^2}{E} - \alpha \tau G' \right) = 0, \quad (40)$$

$$\frac{E^2 H''}{G} - \frac{E^2 H'}{G} \left(\frac{G'}{G} - 2 \frac{E'}{E} - \frac{3}{\tau} - \frac{\alpha}{C_\rho} \frac{\tau G}{E^2} \right) - \frac{E^2}{\tau G} \left(\frac{G'}{G} - 2 \frac{E'}{E} \right) (H - 1) = 0, \quad (41)$$

$$\begin{aligned} \frac{E^2 R_1''}{G} - \frac{E^2 R_1'}{G} \left(\frac{G'}{G} - 2 \frac{E'}{E} - \frac{5}{\tau} - \frac{\alpha}{C_{1\rho}} \frac{\tau G}{E^2} \right) - \frac{E^2 R_1}{\tau G} \left(2 \frac{G'}{G} - 4 \frac{E'}{E} + \frac{C_T}{C_{1\rho}} \frac{\tau G}{E^3} \right) \\ + 2 \frac{C_\rho}{C_{1\rho}} \frac{E^2 H'}{G} \left(2 \frac{(H - 1)}{\tau} + H' \right) = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{E^2 R_2''}{G} - \frac{E^2 R_2'}{G} \left(\frac{G'}{G} - 2 \frac{E'}{E} - \frac{1}{\tau} - \frac{\alpha}{C_{1\rho}} \frac{\tau G}{E^2} \right) - \frac{R_2}{C_{1\rho}} \left(C_T \frac{G}{E} + 2\alpha \right) + 2 \frac{E^2 R_1}{G} \\ + 2 \frac{C_\rho}{C_{1\rho}} (H - 1)^2 \frac{E^2}{G} = 0, \end{aligned} \quad (43)$$

where $\tau = \sqrt{\xi^2 + \eta^2}$.

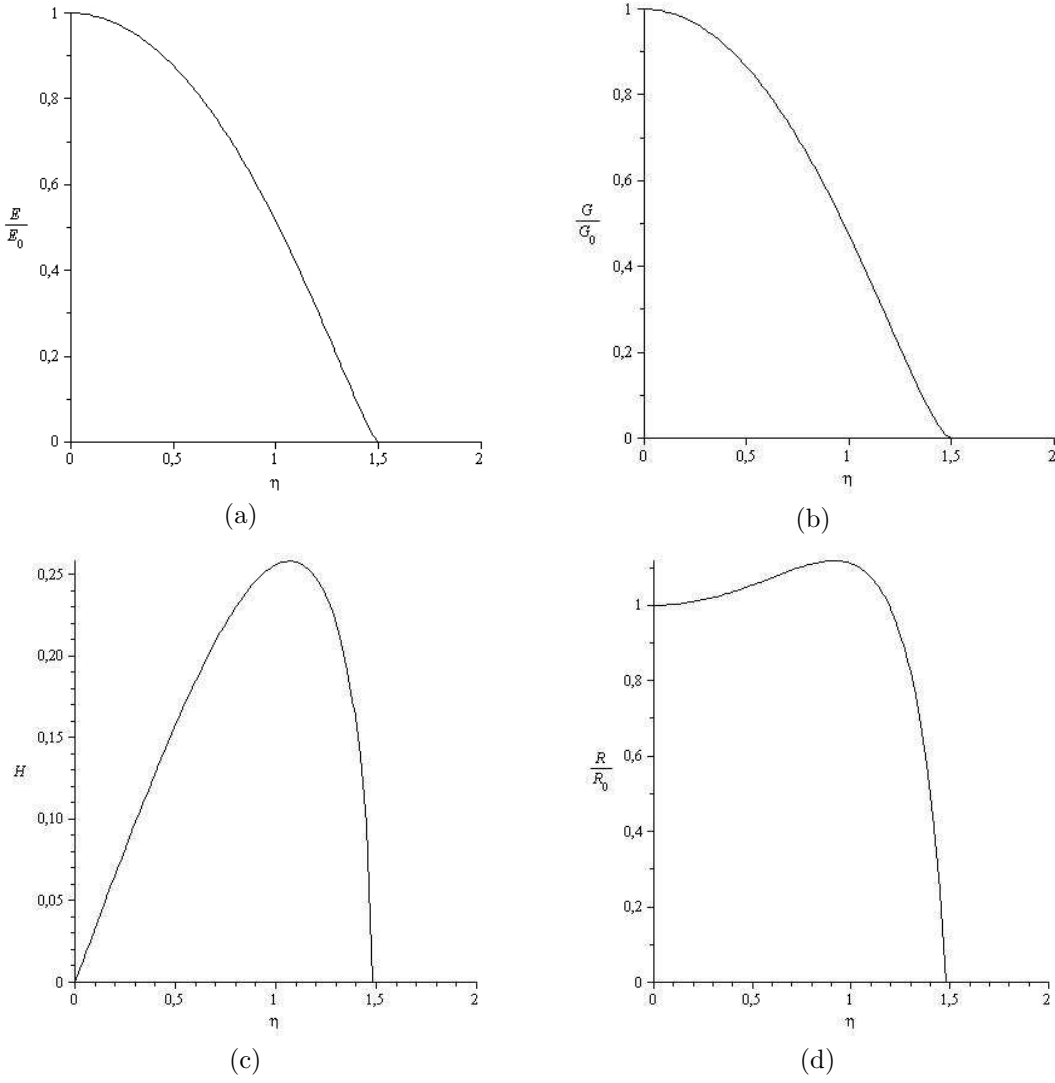


Figure 1. Calculated profiles as $\xi = 0$: (a) normed profile of E , (b) normed profile of G , (c) profile of H , (d) normed profile of R .

5 Calculation results

The system of ODEs (39)–(43) has to satisfy the conditions

$$E' = G' = H'_1 = R'_1 = R'_2 = 0, \quad \tau = 0, \quad (44)$$

$$E = G = H_1 = R_1 = R_2 = 0, \quad \tau \rightarrow \infty. \quad (45)$$

Conditions (44) take into account flow symmetry with respect to the OX axis. The boundary conditions (45) imply that all functions take zero values outside the turbulent wake.

The system of ODEs (39)–(43) satisfying boundary conditions (44), (45) was solved numerically. Additional difficulties are caused by the fact that the coefficients of ODEs have singularities. The problem was solved by the modified shooting method and asymptotic expansion of the solution in the vicinity of the singular point [3, 15]

$$E = c_1(\tau - a)^{10/7} + o(|\tau - a|^{10/7}), \quad G = -\frac{30C_e c_1^2}{7a}(\tau - a)^{13/7} + o(|\tau - a|^{13/7}),$$

$$H = \frac{7C_\rho}{a(7C_\rho - 10C_e)}(\tau - a) + o(|\tau - a|),$$

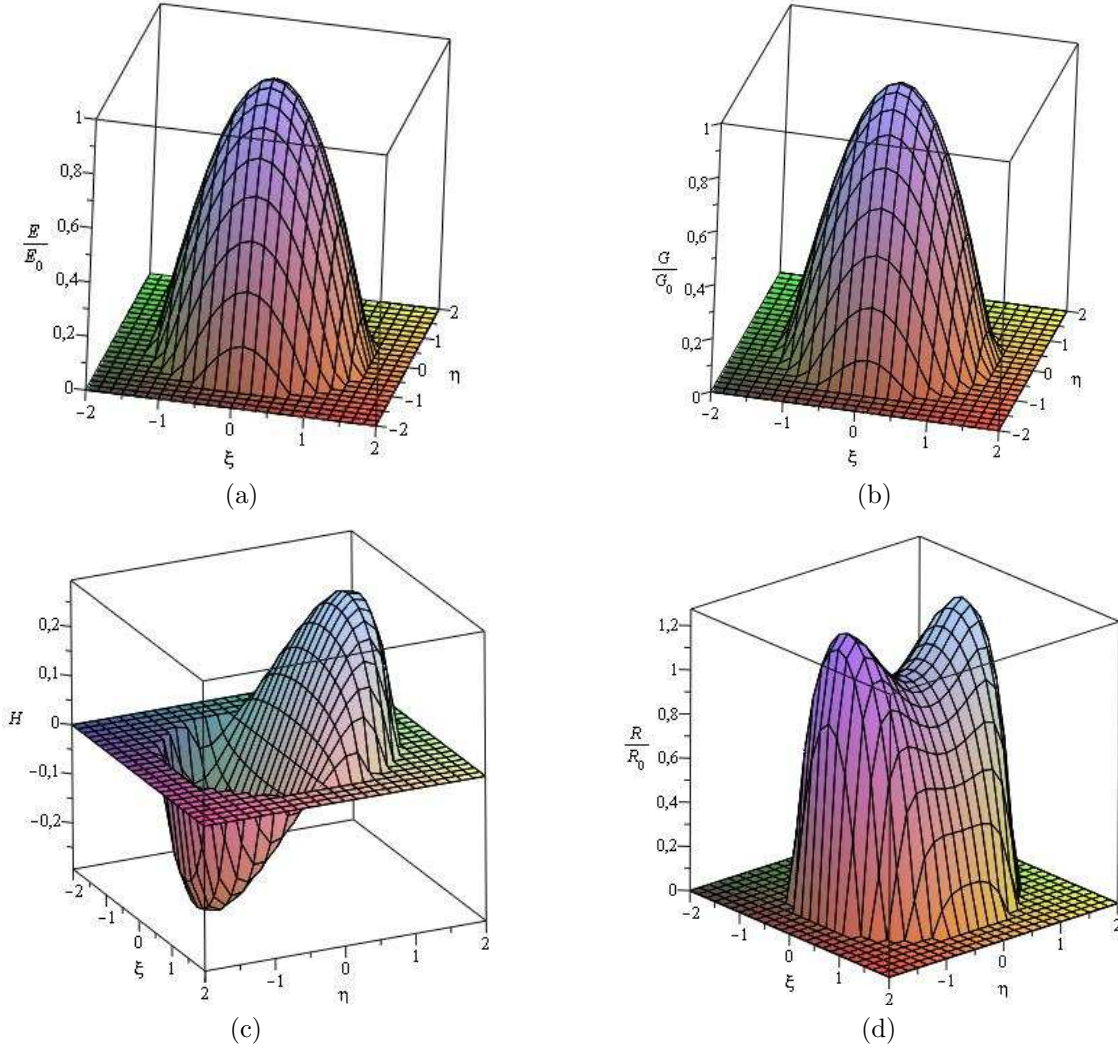


Figure 2. Calculated functions: (a) the function E/E_0 , (b) the function G/G_0 , (c) the function H , (d) the function R/R_0 .

$$R_1 = \frac{49C_\rho^2(7(2a+1)C_\rho - 20aC_e)}{2a^2(7C_\rho - 10C_e)^2(5C_e - 7C_{\rho 1})}(\tau - a)^2 + o(|\tau - a|^2),$$

$$R_2 = \frac{7C_\rho}{2(5C_e - 7C_{\rho 1})}(\tau - a)^2 + o(|\tau - a|^2).$$

The value of α is taken to be 0.23 in accordance with experimental data [9, 17]. The results for the problem solution are illustrated in Figs. 1 and 2. Fig. 1 shows the profiles of the functions E/E_0 , G/G_0 , H , and R/R_0 as $\xi = 0$, where subscript 0 denotes the axial value. The functions E/E_0 , G/G_0 , H , and R/R_0 are plotted in Fig. 2. The functions E/E_0 and G/G_0 are bell-shaped and determine shapes of the normalized turbulent kinetic energy and the normalized kinetic energy dissipation rate respectively. Similarly, H and R/R_0 determine shapes of the average density defect and the normalized density fluctuation variance respectively.

The function $H(0, \eta)$ characterizing the degree of fluid mixing in the turbulent wake is given in Fig. 1c. The maximum value of this function equals 0.258. This is in consistent with the present notions of incomplete fluid mixing in the wakes [23, 24].

In Fig. 3 adapted from [6] the normalized values of the turbulent energy along the wake axis $e_0^{1/2}/U_0 = e(x, 0, 0)/U_0$ are compared with experimental data [17], computational results [9]

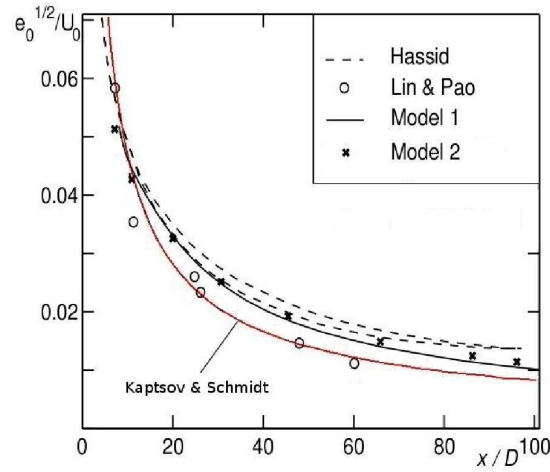


Figure 3. Axial values of the turbulent energy.

and results of numerical simulation based on two semi-empirical turbulence models (Model1 and Model2 in [4, 6]). The coordinate x is normalized by the body diameter D . The results obtained here are in close agreement with Lin and Pao's experimental data.

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